

Parametrizing Del Pezzo surfaces of degree 8 using Lie algebras

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Abstract

For a Del Pezzo surface of degree 8 given over the rationals we decide whether there is a rational parametrization of the surface and construct one in the affirmative case. We define and use the Lie algebra of the surface to reach the aim. The algorithm has been implemented in Magma.

1 Introduction

In this paper we decide whether a given Del Pezzo surface of degree 8 has a rational parametrization over \mathbb{Q} , and find one in the affirmative case. There are two kinds of Del Pezzo surfaces of degree 8: blowups of \mathbb{P}^2 in one point and twists of $\mathbb{P}^1 \times \mathbb{P}^1$. Here we deal with both of them.

The problem is a particular case of a more general one, namely parametrization of surfaces over the rational numbers. There one reduces to several base cases. Except some trivial cases (e.g. \mathbb{P}^2), there are Del Pezzo surfaces of degrees 5 till 9, and conic fibrations. The latter are solved in [10]. Rational parametrization of Del Pezzo surfaces of degree 5 is discussed in [11], and of degree 7 can be found in e.g. [7]. Parametrization of degree 9 is solved in [3]. Hence the last unsolved class are Del Pezzo surfaces of degree 6.

For Del Pezzo surfaces of degree 6 the Hasse principle holds (cf [7]). For a given prime p we can decide (e.g. using Magma [1]) whether there is a point

on the surface over the local field \mathbb{Q}_p . But there still remains the problem of finding a finite set of “bad primes” and finding a rational point provided that we have points over local fields.

Here we use another approach. Similarly as in [3], the parametrization problem is reduced to a problem concerning Lie algebras and their representations. The Lie algebra approach appears to be preferable even in situations where also other methods are known, such as the parametrization problem of blowups of the plane in one point, see [7].

The paper is structured as follows. In section 2, we introduce the Lie algebra of a variety and give an algorithm for computing it. Section 3 is dealing with “identification problems”; the main focus will be to reduce the problem of identifying a variety (i.e. constructing an isomorphism if exists) to the problem of identifying a Lie algebra. This will solve some instances of our parametrization problem of Del Pezzo surfaces of degree 8. The remaining instances are dealt with in section 4.

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2 The Lie Algebra of a Variety

In this section, we define the Lie algebra of a variety and give a method for computing it.

Throughout, we assume that F is a field of characteristic zero. We are mostly interested in the case $F = \mathbb{Q}$, but the algorithm to be described works equally well for number fields, and there is one step where field extensions are needed (see section 4).

Let X be a projective variety over F . We denote the group of its automorphisms by $\text{Aut}(X)$. The first idea to define the Lie algebra of X would be to take the tangent space of $\text{Aut}(X)$ at the identity, but this does not work in general because $\text{Aut}(X)$ need not be an algebraic group. Hence we introduce

$$\text{Aut}_0(X) = \{\varphi \in \text{Aut}(X) \mid \varphi \text{ acts trivially on } \text{Pic}(X)\}.$$

The advantage of working with $\text{Aut}_0(X)$ rather than with $\text{Aut}(X)$ become clear by this

Theorem 2.1 *The group $\text{Aut}_0(X)$ is an algebraic group over F . For any*

very ample divisor D of X , there is a faithful representation of $\text{Aut}_0(X)$ into $\text{PGL}_{n+1}(F)$, where $n := \dim(D)$.

Proof. Let $i : X \hookrightarrow \mathbb{P}^n$ be the rational map associated to D . It is an embedding since D is very ample. Suppose that $\varphi \in \text{Aut}_0(X)$. Then the pullback $\varphi^* : \text{Div}(X) \rightarrow \text{Div}(X)$ transforms the complete linear system $|D|$ into itself. Routine calculation shows that this transformation is projective and its dual is an extension of φ to the ambient projective space \mathbb{P}^n . Clearly, X contains $n+2$ points in general position (not necessarily defined over F), hence any projective transformation leaving X pointwise fixed is the identity, and it follows that the representation of $\text{Aut}_0(X)$ into $\text{PGL}_{n+1}(F)$ is faithful.

If we choose a different very ample divisor D_1 , then the composition of homomorphisms to and from $\text{Aut}_0(X)$ is algebraic (i.e. it is a regular function). Hence the algebraic structure of $\text{Aut}_0(X)$ does not depend on the choice of D . \square

For convenience, we would prefer a representation in $\text{GL}_{n+1}(F)$ rather than in $\text{PGL}_{n+1}(F)$. This is not always possible, for instance, it is not clear how to embed $\text{Aut}_0(\mathbb{P}^1)$ into $\text{GL}_2(\mathbb{Q})$. But after passing to Lie algebras the situation is much easier: we just add a direct one-dimensional abelian summand in order to compensate the difference between $\mathfrak{gl}_{n+1}(F)$ and $\mathfrak{sl}_{n+1}(F)$, the Lie algebra of $\text{PGL}_{n+1}(F)$.

Definition 2.2 *Let X be a projective variety over F . We define $L_0(X, F)$ as the Lie algebra of the algebraic group $\text{Aut}_0(X)$, and $L(X, F)$ as the direct sum of $L_0(X, F)$ and the abelian one-dimensional Lie algebra C .*

We also write $L_0(X)$ and $L(X)$ as shorthands, if there is no ambiguity of the field. We refer to $L_0(X)$ as the Lie algebra of the variety X .

Here is a theorem that can be used for the computation of $L(X, F)$.

Theorem 2.3 *Let X be a projective variety such that $\text{Pic}(X)$ is discrete. Let $D \in \text{Div}(X)$ be a very ample divisor, and let $n := \dim(D) + 1$. Let $i : X \rightarrow \mathbb{P}^{n-1}$ be the associated embedding. Let $\text{Aut}_i(X) \subset \text{GL}_n(F)$ be the group of all invertible linear maps whose projectivization maps $i(X)$ into itself. Then $L(X, F)$ is the Lie algebra of $\text{Aut}_i(X)$.*

Proof. Note that $\text{Aut}_i(X)$ is an algebraic group, because it can be given by polynomial equations, namely $g \in \text{Aut}_i(X)$ if and only if $f_i(gp) = 0$ for all $p \in X$ and all i such that f_i 's generate the vanishing ideal of the

embedded variety $i(X)$. The multiplicative group Z of scalar matrices is an algebraic subgroup in the center of $\text{Aut}_i(X)$. The quotient group $\text{Aut}_i(X)/Z$ is an algebraic group of automorphisms of X containing $\text{Aut}_0(X)$. Because the Picard group is discrete, the connected component of the identity of $\text{Aut}_i(X)/Z$ leaves it pointwise fixed, hence it is contained in $\text{Aut}_0(X)$. It follows that $\text{Aut}_i(X)/Z$ and $\text{Aut}_0(X)$ have the same Lie algebra, namely $L_0(X, F)$.

Because Z is contained in the center of $\text{Aut}_i(X)$, its Lie algebra C is contained in the center of the Lie algebra of $\text{Aut}_i(X)$. It follows that C is a direct summand, and the co-summand is the Lie algebra of the quotient. \square

Note that by this construction, $L(X, F)$ is a Lie algebra of matrices, so the construction gives not only $L(X, F)$ but also a representation $L(X, F) \hookrightarrow \mathfrak{gl}_n(F)$; of course, the representation depends on the embedding i .

Example 2.4 Let $r > 0$. Let $X = \mathbb{P}^r$. Then every automorphism fixes the Picard group, which is isomorphic to \mathbb{Z} . So we have $\text{Aut}(X) = \text{Aut}_0(X) = \text{PGL}_{r+1}(F)$, and $L_0(X, F) = \mathfrak{sl}_{r+1}(F)$, and $L(X, F) = \mathfrak{gl}_{r+1}(F)$.

Let $d > 0$. Let D be a divisor of degree d . Then $n = \dim(D) = \binom{r+d+1}{d} - 1$, and the associated map $i : X \rightarrow \mathbb{P}^n$ is the d -uple embedding. The group $\text{Aut}_i(X)$ is the d -th symmetric power of $\text{GL}_{r+1}(F)$, and its Lie algebra is the representation of $\mathfrak{gl}_{r+1}(F)$ by d -th symmetric powers.

The paper [3] contains the following converse of Example 2.4: if X is a twist of \mathbb{P}^r and $L(X, F) \cong \mathfrak{gl}_{r+1}(F)$, then $X \cong \mathbb{P}^r$. The problem of constructing the isomorphism from \mathbb{P}^r to X can be reduced to the construction of a Lie algebra isomorphism from $\mathfrak{gl}_{r+1}(F)$ to $L(X, F)$. In section 3.2, we will prove a similar result for twists of $\mathbb{P}^1 \times \mathbb{P}^1$.

In the applications we are interested, the ideal of the variety X can be given by quadratic equations. This is equivalent to D having the property N_1 (see [9]). In this case, there is a particularly easy way of computing its Lie algebra.

Theorem 2.5 *Let $X \subset \mathbb{P}^n$ be an embedded projective variety. Assume that the ideal of X is generated by quadrics. Write all of these quadratic equations as $p^T A p$, where A is a symmetric matrix of size $(n+1) \times (n+1)$. Let I be the linear space generated by these matrices.*

Then the Lie algebra $L(X, F)$ is the matrix algebra

$$\{x \in \mathfrak{gl}_{n+1}(F) \mid x^T A + A x \in I \text{ for all } A \in I\}.$$

Proof. Let i be the embedding of X . We have

$$\mathrm{Aut}_i(X) = \{g \in \mathrm{GL}_{n+1}(F) \mid g^T Ag \in I \text{ for all } A \in I\}.$$

Let W denote the vector space of $(n+1) \times (n+1)$ -matrices over F . We have a rational representation $\rho : \mathrm{GL}_{n+1}(F) \rightarrow \mathrm{GL}(W)$ given by $\rho(g)(A) = g^T Ag$. Then $\mathrm{Aut}_i(X)$ is the group of all $g \in \mathrm{GL}_{n+1}(F)$ such that $\rho(g)I = I$. By [2], Corollary 1 to Theorem 1, Chapter III, No 9, the Lie algebra of $\mathrm{Aut}_i(X)$ consists of all $x \in \mathfrak{gl}_{n+1}(F)$ such that $(d\rho)(x)(I) \subset I$. Now $(d\rho)(x)(A) = x^T A + Ax$. \square

Of course, it is sufficient to collect all conditions for A in a fixed basis of I . Hence $L(X, F)$ can be computed by linear algebra.

3 Identification Problems

In this section we treat some special instances and subproblems of the parametrization problem for Del Pezzo surfaces of degree 8, of the following type: given a variety X , decide whether it is equivalent to a fixed variety Y ; and if yes, construct an isomorphism from Y to X . We call this type of problem the *identification problem* for Y .

We solve it for $Y = \mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^1$, and the blowup of \mathbb{P}^2 at a single point. We denote this blowup variety by \mathbb{Y} . (For $Y = \mathbb{P}^2$, the problem was solved in [3].) In all these cases, the existence part of the identification problem is not difficult to solve when F is algebraically closed. So we may assume that X is a twist of Y , i.e. that X is isomorphic to Y over the algebraic closure of F .

In all the cases above, the anticanonical divisor $-K$ is very ample. If X and Y are isomorphic, then the anticanonical images $i_{-K}(X)$ and $i_{-K}(Y)$ are projectively isomorphic, because any isomorphism induces a linear isomorphism between the spaces of global sections of the two anticanonical line bundles (see also [3]). The problem of deciding whether two embedded projective varieties are projectively isomorphic, and to construct a projective transformation if exists, will be called the *embedded identification problem*.

A necessary condition for X being isomorphic to Y is that $L_0(X)$ is isomorphic to $L_0(Y)$. If both X and Y are anticanonically embedded, the isomorphism $Y \rightarrow X$ is described by $p \mapsto Mp$ for some matrix $M \in \mathrm{GL}_{n+1}(F)$, where $n = \dim(-K_X) = \dim(-K_Y)$. Then we also have a Lie algebra isomorphism $\nu : L_0(Y) \rightarrow L_0(X)$ given by $\nu(x) = MxM^{-1}$ for the same matrix

M . The matrix M defines an isomorphism of the $L_0(Y)$ -modules given by the inclusion $L_0(Y) \hookrightarrow \mathfrak{gl}_{n+1}(F)$ and by its composition with the Lie algebra isomorphism ν .

Given X , we claim that the embedded identification problem for Y can be solved by the following algorithm (assuming Y is one of \mathbb{P}^1 , $\mathbb{P}^1 \times \mathbb{P}^1$, or \mathbb{Y}):

1. Compute $L_0(X)$ and solve the *Lie algebra identification problem* for $L_0(Y)$; i.e., construct a Lie algebra isomorphism ν if exists. Otherwise, X and Y are not isomorphic.
2. Construct an isomorphism M between the $L_0(Y)$ -modules defined by the inclusion $L_0(Y) \hookrightarrow \mathfrak{gl}_{n+1}(F)$ and by the composition with ν . If the modules are not isomorphic, then X and Y are not projectively equivalent.
3. Check if M transforms Y to X . If yes, we have found the isomorphism. Otherwise, X and Y are not projectively equivalent.

Methods for solving the Lie algebra identification problem (step 1) and for solving the module identification problem (step 2) will be explained in the subsequent subsections.

The correctness of the algorithm follows from the following statements.

- Assume Y and X are projectively equivalent via some matrix M , and $\nu : L_0(Y) \rightarrow L_0(X)$ is a Lie algebra isomorphism. Then conjugation by M is another isomorphism from $L_0(X)$ to $L_0(Y)$ (cf. [3], Proposition 3.4). Composing these two, we get a Lie algebra automorphism of $L_0(Y)$. By Lemma 3.2 below, this automorphism is equal to the conjugation by a matrix $N \in \text{Aut}_i(Y)$. Then ν is equal to conjugation by NM , and NM is an isomorphism of the two $L_0(Y)$ -modules in step 2. In particular, these $L_0(Y)$ -modules are isomorphic.
- Assume Y and X are projectively equivalent via some matrix U , and conjugation by $M \in \text{GL}_{n+1}(F)$ is a homomorphism from $L_0(X)$ into $L_0(Y)$ (this is equivalent to M being a module isomorphism as computed in step 2). Then we claim that M maps Y to X . For this we may assume that the ground field is algebraically closed. Indeed, by Theorem 2.5, $L_0(Y, \overline{F}) = L_0(Y, F) \otimes \overline{F}$, and similarly for $L_0(X, \overline{F})$. Furthermore, conjugation by M is also a homomorphism of $L_0(X, \overline{F})$

into $L_0(Y, \overline{F})$. Note that conjugation by $U^{-1}M$ is an automorphism of $L_0(Y)$. By Lemma 3.2 below, $U^{-1}M$ transforms Y to itself, hence $M = U(U^{-1}M)$ transforms Y to X .

Lemma 3.1 *Let $G \subset \mathrm{GL}(V)$, $H \subset \mathrm{GL}(W)$ be algebraic groups, where V and W are vector spaces over an algebraically closed field. Assume that G and H have the same dimension, and the same number of connected components. Let $\sigma : G \rightarrow H$ be an injective rational representation of G . Then σ is surjective.*

Proof. Let G^0, H^0 be the connected components of the identity. Then $\sigma(G^0)$ is connected as well. Hence $\sigma(G^0) \subset H^0$. Furthermore, by [2], Corollary 1 to Proposition 2, Chapter II, §7, $\sigma(G^0)$ is an algebraic subgroup of H . Furthermore, $\dim \sigma(G^0) = \dim G^0$ (this follows for example from [2], Corollary to Proposition 8, Chapter II, §6). Hence $\sigma(G^0)$ has finite index in H^0 ([2], Proposition 4, Chapter II, §6). Since H^0 is the unique irreducible algebraic subgroup of H of finite index, we conclude that $\sigma(G^0) = H^0$. Let x_1G^0, \dots, x_tG^0 be the irreducible components of G . Then $\sigma(x_iG^0) = \sigma(x_i)H^0$. Hence the $\sigma(x_i)H^0$ are irreducible components of H . Since H has the same number of such components as G , we conclude that σ is surjective. \square

Lemma 3.2 *Suppose that the ground field is algebraically closed, and that the centre of $L_0(Y)$ is 0. Let $Z = \{\lambda I_{n+1}\}$, where $I_{n+1} \in \mathrm{GL}_{n+1}(F)$ is the identity. If $\mathrm{Aut}_i(Y)/Z$ and $\mathrm{Aut}(L_0(Y))$ have the same dimension and number of connected components, then any automorphism of $L_0(Y)$ is given as conjugation by an element of $\mathrm{Aut}_i(Y)$.*

Proof. Let $\mathrm{Ad} : \mathrm{Aut}_i(Y) \rightarrow \mathrm{Aut}(L(Y))$ be given by $\mathrm{Ad}(g)(x) = gxg^{-1}$. From [2], Proposition 12, Chapter III, No 9, it follows that the Lie algebra of the kernel of Ad is equal to the centre of $L(Y)$, which is spanned by I_{n+1} . Hence $\ker \mathrm{Ad} = Z$. Therefore the induced homomorphism $\mathrm{Ad} : \mathrm{Aut}_i(Y)/Z \rightarrow \mathrm{Aut}(L_0(Y))$ is injective. Lemma 3.1 implies that Ad is surjective, hence the second statement follows. \square

For the three varieties that we consider we will check the hypothesis of Lemma 3.2 in separate subsections.

Remark 3.3 We remark that the hypothesis of Lemma 3.2 does not hold for $Y = \mathbb{P}^2$. In order to construct the module isomorphism, one sometimes has

to correct the Lie algebra isomorphism by an outer automorphism of $L_0(Y)$. See [3] for details.

For each of the three choices of Y , namely \mathbb{P}^1 , $\mathbb{P}^1 \times \mathbb{P}^1$, and \mathbb{Y} , we still have to do three things:

1. solve the Lie algebra identification problem;
2. solve the Lie module identification problem;
3. check the hypothesis of Lemma 3.2.

Note that 2., the Lie module identification problem, is a just linear problem, because the sought matrix M as a generic solution of the system of equations $M(xv) = x(Mv)$ for all $x \in L_0(Y)$ and $v \in F^{n+1}$. A generic solution will be non-singular. However, if the Lie algebra is split semisimple, then we can use the theory of weight vectors to find a module isomorphism. This is more efficient then solving the system of linear equations above.

3.1 Identifying \mathbb{P}^1

Solutions for the identification problem for \mathbb{P}^1 are well-known. Nevertheless, we want to describe how solve it by Lie algebras because of two reasons: first, it is the simplest possible example where the method works, and second, we have to solve the corresponding Lie algebra identification problem anyway at another place.

Using the anticanonical embedding, we can reduce to the embedded identification problem of the parabola with equation $y_0y_2 - y_1^2 = 0$ in \mathbb{P}^2 . The embedded twists of the parabola are exactly the nonsingular conics, and such a twist is projectively isomorphic to the parabola iff it has a point defined over F . Hence we see that our problem is equivalent to deciding whether a given ternary quadratic form is isotropic, i.e. has a nontrivial solution over F ; constructing an explicit isomorphism is possible when we have such an explicit solution. We will see that the Lie algebra method reduces to the same problem.

Identification of the Lie algebra. We have that $\text{Aut}_i(Y)/Z$ (where Z consists of the scalar matrices) is isomorphic to $\text{PGL}_2(F)$. Therefore $L_0(Y) \cong \mathfrak{sl}_2$. The twists of \mathfrak{sl}_2 are the semisimple Lie algebras of dimension 3, because dimension and semisimplicity do not change under field extension, and over

algebraically closed fields \mathfrak{sl}_2 is the only semisimple Lie algebra of dimension 3. For checking semisimplicity, we can use Cartan's criterion saying that this is equivalent to the Killing form being non-degenerate. Finally, here is a proposition that allows to identify a twist.

Proposition 3.4 *Let L be a semisimple Lie algebra of dimension 3. Then L is isomorphic to \mathfrak{sl}_2 iff its Killing form is isotropic.*

Proof. It is easy to check that the Killing form of \mathfrak{sl}_2 is isotropic, hence “only if” is clear.

Conversely, let $a \in L$ be a non-zero isotropic element. Note first, that for any nonzero b in a twist of \mathfrak{sl}_2 we have that the trace of $\text{ad}(b)$ equals 0 and also that the kernel of $\text{ad}(b)$ is generated by b , for if it was two-dimensional, then $[L, L]$ would be a nontrivial ideal in L .

Let e_1, e_2, e_3 be the eigenvalues of $\text{ad}(a)$, so $e_1 + e_2 + e_3 = 0$. Since a is an isotropic element of the Killing form, we have also $e_1^2 + e_2^2 + e_3^2 = 0$. One of eigenvalues is zero and so we get that all e_i vanish. So $\text{ad}(a)$ is nilpotent and hence there exists an element b such that $[a, b] = a$. Then $\text{ad}(b)$ has an eigenvalue of -1 , hence it is split semisimple and b generates a split Cartan subalgebra H . When we have H , an isomorphism to \mathfrak{sl}_2 can be constructed explicitly (see [4]). \square

Solving ternary quadratic forms can be done over number fields in Magma: first, we check for local solvability at all primes dividing the Hessian. If the form is everywhere solvable, then there is a solution in F by the Hasse principle (see [8]). The construction of the solution can be reduced to solving a norm equation of a quadratic extension of F . If $F = \mathbb{Q}$, then we use faster algorithms for finding a rational point on a plane conic.

Identification of the Lie module. We show that the \mathfrak{sl}_2 -module given by the isomorphism $\mathfrak{sl}_2 \rightarrow L_0(Y)$ is irreducible, of highest weight (2). Let V be the 2-dimensional vector space over F with basis $\{v_0, v_1\}$. Let $W = \text{Sym}^2(V)$ with the basis $\{v_0^2, 2v_0v_1, v_1^2\}$. Let $\varphi : V \rightarrow W$ be defined by $\varphi(v) = v^2$. We write the coordinates of an element of W with respect to the basis above. Then the image of the induced map $\varphi : \mathbb{P}(V) \rightarrow \mathbb{P}(W)$ is exactly Y .

Let $\text{GL}_2(F)$ act naturally on V , i.e. the vector with the coordinates (s, t) is mapped by $g = (g_{ij})_{i,j=0}^1 \in \text{GL}_2(F)$ to the vector with the coordinates $(g_{00}s + g_{01}t, g_{10}s + g_{11}t)$. This leads to the action of $\text{GL}_2(F)$ on W by $g \cdot vv' = (gv)(gv')$, for $v, v' \in V$. By writing the matrix of elements of $\text{GL}_2(F)$ with respect to the basis above we get a representation $\rho : \text{GL}_2(F) \rightarrow \text{GL}_3(F)$.

We have $g \cdot \varphi(v) = \varphi(g \cdot v)$, and hence $\varphi(V)$ is fixed under the action of $\mathrm{GL}_2(F)$ on W . We have further $Y = \varphi(\mathbb{P}(V))$, therefore $\rho(\mathrm{GL}_2(F)) \subseteq \mathrm{Aut}(\varphi(V)) = \mathrm{Aut}_i(Y)$. The kernel of ρ consists of two matrices, $\pm I_2$, the identity in $\mathrm{GL}_2(F)$. The conclusion is that the $\mathrm{GL}_2(F)$ -module given by ρ is isomorphic to $\mathrm{Sym}^2(V)$. Hence the same holds for the corresponding modules of the Lie algebras.

Using highest weight vectors, we can construct a module isomorphism. This isomorphism is unique up to scalar multiplication, because the module is irreducible.

Checking the hypothesis of Lemma 3.2. We have that $L_0(Y)$ is isomorphic to $\mathfrak{sl}_2(F)$ and hence its centre is 0. As in Lemma 3.2, let Z be the subgroup of $\mathrm{Aut}_i(Y)$ consisting of scalar multiples of the identity. Then both groups $\mathrm{Aut}_i(Y)/Z$ and $\mathrm{Aut}(\mathfrak{sl}_2)$ are isomorphic to $\mathrm{PGL}_2(F)$ (for $\mathrm{Aut}(\mathfrak{sl}_2)$ see [6], Chapter IX, Theorem 5), hence both have dimension 3 and are connected.

□

3.2 Identifying $\mathbb{P}^1 \times \mathbb{P}^1$

Because $\mathbb{P}^1 \times \mathbb{P}^1$ is a Del Pezzo surface of degree 8 (in its anticanonical embedding), this identification problem is an instance of our parametrization problem.

Identification of the Lie algebra. Let Z be the subgroup of $\mathrm{Aut}_i(Y)$ consisting of the scalar matrices. Then $\mathrm{Aut}_i(Y)/Z \cong \mathrm{PGL}_2(F) \times \mathrm{PGL}_2(F) \ltimes \mathbb{Z}/2\mathbb{Z}$. Hence $L_0(Y) \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. Let L be a given Lie algebra, then to decide isomorphism with $L_0(Y)$ we do the following. We check whether it is semisimple, decompose into its simple components, and solve the identification problem for \mathfrak{sl}_2 (as in subsection 3.1) for the two components. If the number of components differs from 2, then L is not isomorphic to $L_0(Y)$. An algorithm for decomposing semisimple Lie algebras can be found in [4].

Identification of the Lie module. The anticanonical embedding $i : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^8$ is given by

$$(s_0:s_1; t_0:t_1) \mapsto (s_0^2 t_0^2 : s_0^2 t_0 t_1 : s_0^2 t_1^2 : s_0 s_1 t_0^2 : s_0 s_1 t_0 t_1 : s_0 s_1 t_1^2 : s_1^2 t_0^2 : s_1^2 t_0 t_1 : s_1^2 t_1^2).$$

In this case the $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ -module given by the isomorphism $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \rightarrow L_0(Y)$ is irreducible of highest weight $(2, 2)$. This can be shown as it was done for the \mathbb{P}^1 case (Section 3.1). In this case we use the map $\varphi : V \times W \rightarrow$

$\text{Sym}^2(V) \otimes \text{Sym}^2(W)$, where V, W are 2-dimensional. Then the projectivization of $\varphi(V \times W)$ is equal to Y . Here the first direct summand of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ acts on $\text{Sym}^2(V)$ and the second summand on $\text{Sym}^2(W)$. Hence the full algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ acts on the tensor product. This means that the module is irreducible and of highest weight $(2, 2)$.

Hence we can decide module equivalence by checking irreducibility and computing the highest weight. In the affirmative case, we can again construct a module isomorphism by using highest weight vectors. It is unique up to scalar multiplication, as in the previous case.

Checking the hypothesis of Lemma 3.2. Because Y is anticanonically embedded, we have $\text{Aut}_i(Y)/Z = \text{Aut}(Y)$, which has dimension 6 because its Lie algebra has dimension 6. The normal subgroup $\text{Aut}_0(Y)$ has the same dimension, but it is a proper subgroup because the automorphism interchanging the two product factors \mathbb{P}^1 does not preserve classes. Hence $\text{Aut}_i(Y)/Z$ has at least 2 components. On the other hand, the group of automorphism of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is a semidirect product of the group of inner automorphism and the finite group of “diagram automorphisms” (see [6], §IX.4). The group of inner automorphism is connected of dimension 6, and the group of diagram automorphisms is $\mathbb{Z}/2\mathbb{Z}$ as the Dynkin diagram consists of two nodes and no edges. Hence $\text{Aut}(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2)$ has dimension 6 and 2 connected components. The centre of $L_0(Y)$ is 0. So as seen in the proof of Lemma 3.2, the homomorphism $\text{Ad} : \text{Aut}_i(Y)/Z \rightarrow \text{Aut}(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2)$ is injective. Therefore $\text{Aut}_i(Y)/Z$ has exactly two components. \square

Timings. We implemented the algorithm in Magma. The examples were constructed as follows. We took the canonical $\mathbb{P}^1 \times \mathbb{P}^1$ in \mathbb{P}^8 given by 20 binomials. Then we generated a 9×9 matrix containing random integer numbers with absolute values up to a given maximal number (this is written in the first column of Table 1). We used this matrix as the matrix of a linear transformation of projective space obtaining so a different system of implicit equations. For a “small” perturbation, almost the whole time is spent for finding the Lie algebra of the surface. As the coefficients of the linear transformation grow, finding a rational point on the conic starts to play the main role in the time complexity.

perturb	eqns max	LA size	prm size	time	LA time	conic time
1	4	11	18	4.56	4.49	0.00
5	73	47	70	21.93	21.66	0.03
10	255	55	84	28.46	28.11	0.09
50	5026	84	130	48.75	48.15	0.22
100	25304	111	166	61.02	60.15	0.34
300	225440	134	200	75.86	73.00	2.14
	208199	136	204	89.52	73.15	15.77
400	335499	143	213	77.99	76.31	0.93
	418185	141	210	152.21	77.91	73.56
	545728	140	208	482.69	74.50	407.53
500	720193	147	222	91.11	82.24	8.10
	525179	145	216	80.95	78.91	1.29
	546787	143	218	176.13	78.51	96.96

- perturb – maximum entry allowed in perturbation matrix,
eqns max – the maximal absolute value of the coefficients in the implicit equations,
LA size – the maximal length of the numerator/denominator of the structure constants of the Lie algebra,
prm size – the maximal length of the numerator/denominator of the coefficients in the parametrization,
time – the time (in sec) needed for parametrizing,
LA time – the time (in sec) needed for finding the Lie algebra (is a part of “time” in the previous column).
conic time – the time (in sec) needed for finding rational points on two conics constructed to identify two summands $\mathfrak{sl}_2(\mathbb{Q})$ (is a part of “time”).

Table 1: Parametrizing $\mathbb{P}^1 \times \mathbb{P}^1$.

3.3 Identifying \mathbb{Y}

Because \mathbb{Y} , in its anticanonical embedding, is also a Del Pezzo surface of degree 8, this identification problem is another instance of our parametrization problem.

Identification of the Lie algebra.

Every automorphism of \mathbb{Y} leaves the exceptional line invariant, so $\text{Aut}(\mathbb{Y})$ is isomorphic to the subgroup $\text{Aut}(\mathbb{P}^2) = \text{PGL}_3(F)$ fixing the point $(1:0:0)$.

The whole group leaves $\text{Pic}(\mathbb{Y})$ invariant, so $\text{Aut}_0(\mathbb{Y}) = \text{Aut}(\mathbb{Y})$. Its Lie algebra is isomorphic to

$$L_0(\mathbb{Y}) = \left\{ \begin{pmatrix} 2a & b_1 & b_2 \\ 0 & -a + c_1 & c_2 \\ 0 & c_3 & -a - c_1 \end{pmatrix} \mid a, b_1, b_2, c_1, c_2, c_3 \in F \right\}.$$

Here is a useful characterization of this Lie algebra.

Proposition 3.5 *Let L be a Lie algebra. Then L is isomorphic to $L_0(\mathbb{Y})$ iff it has a 2-dimensional ideal I which is abelian as a subalgebra, and a 4-dimensional subalgebra S isomorphic to \mathfrak{gl}_2 , such that the adjoint action of S on I is faithful.*

Proof. “Only if”: for $L = L_0(\mathbb{Y})$, we take I as the ideal defined by $a = c_1 = c_2 = c_3 = 0$, and S as the subalgebra defined by $b_1 = b_2 = 0$.

“If”: the Lie algebra $\text{Der}(I)$ is isomorphic to \mathfrak{gl}_2 . Because any injective homomorphism from \mathfrak{gl}_2 to itself is an automorphism, the action of S on I is determined up to isomorphism. Therefore L is isomorphic to the semidirect sum $I \rtimes S$ with respect to this action. \square

To solve the identification problem for $L_0(\mathbb{Y})$ with input L , we can proceed as follows.

1. Take I as the nilradical of L . If this is not two-dimensional abelian, then L is not isomorphic to $L_0(\mathbb{Y})$.
2. Take S as the normalizer of a Levi subalgebra of L . If $\dim(S) \neq 4$, then L is not isomorphic to $L_0(\mathbb{Y})$.
3. Check if the adjoint action of S on I is faithful. If not, then L is not isomorphic to $L_0(\mathbb{Y})$. If yes, one can construct an isomorphism using the construction of semidirect sums.

For checking the correctness of the construction, it suffices to check it for $L_0(\mathbb{Y})$; and this is a routine calculation.

Identification of the Lie module.

Let K be a Levi subalgebra of $L_0(\mathbb{Y})$ (for instance the subalgebra defined by $a = b_1 = b_2 = 0$). The given $L_0(\mathbb{Y})$ -module $W = F^9$ is also an K -module. We analyze this module by a similar method as the one we used in Section 3.1.

Let V be a 3-dimensional vector space with basis v_0, v_1, v_2 . Consider the symmetric power $\text{Sym}^3(V)$ with the basis $v_0^3, 3v_0^2v_1, 3v_0^2v_2, 3v_0v_1^2, 6v_0v_1v_2, 3v_0v_2^2, v_1^3, 3v_1^2v_2, 3v_1v_2^2, v_2^3$. Let $\varphi' : V \rightarrow \text{Sym}^3(V)$ be given by $\varphi'(v) = v^3$.

Let $G = \text{GL}_3(\mathbb{Q})$ act naturally on V . Let $\rho'(g)$ be the matrix describing the action of $g \in G$ on $\text{Sym}^3(V)$ with respect to the basis above.

Let U be the subspace of $\text{Sym}^3(V)$ spanned by v_0^3 . Let $\pi : \text{Sym}^3(V) \rightarrow \text{Sym}^3(V)/U$ be the projection discarding the coordinate at v_0^3 , and set $\varphi = \pi \circ \varphi'$. Then \mathbb{Y} is the projectivization of $\varphi(V)$. Then $\text{Aut}(\mathbb{Y}) = \text{Stab}_G(U)$.

Lemma 3.6 *As a K -module W decomposes as a direct sum $W = W_2 \oplus W_3 \oplus W_4$, where W_i is an i -dimensional irreducible K -module. The elements of the nilradical I carry W_4 to W_3 and W_3 to W_2 .*

Proof. When restricting to the Levi subalgebra K , the module $\text{Sym}^3(V)$ (see the discussion before the Lemma) becomes an \mathfrak{sl}_2 -module and as such decomposes as a sum of four irreducible modules: $W_1 = U$, W_2 is the module spanned by $3v_0^2v_1, 3v_0^2v_2$ and isomorphic to the natural \mathfrak{sl}_2 -module, W_3 is spanned by $3v_0v_1^2, 6v_0v_1v_2, 3v_0v_2^2$ and isomorphic to $\text{Sym}^2(F^2)$, and lastly W_4 is spanned by $v_1^3, 3v_1^2v_2, 3v_1v_2^2, v_2^3$ and isomorphic to $\text{Sym}^3(F^2)$. Then W as \mathfrak{sl}_2 -module decomposes into the sum $W_2 \oplus W_3 \oplus W_4$.

To prove the last assertion of the Lemma, let $b \in I$, $b = b_1e_{12} + b_2e_{13}$, where e_{ij} is the matrix with 1 on the position (i, j) and 0 elsewhere. So if $w \in W_4$ is a basis vector, $w = v_1^i v_2^{3-i}$, then $b \cdot w \in \langle v_0 v_1^{i-1} v_2^{3-i}, v_0 v_1^i v_2^{2-i} \rangle \subset W_3$. Similarly for $w \in W_3$ one gets $b \cdot w \in W_2$. \square

Let $f : W \rightarrow W$ be an isomorphism of $L_0(\mathbb{Y})$ -modules. Then f restricted to W_i is multiplication by a scalar λ_i . Let $b = e_{12} \in I$, and $w_4 = v_1^3 \in W_4$. Then $b \cdot v_1 = v_0$, hence $b \cdot w_4 = 3v_0v_1^2 \in W_3$. Hence $f(b \cdot w_4) = \lambda_3 b \cdot w_4$. On the other hand, $f(b \cdot w_4) = b \cdot f(w_4) = \lambda_4 b \cdot w_4$. We infer that $\lambda_4 = \lambda_3$. In the same way we find that $\lambda_3 = \lambda_2$, so that f is multiplication by a scalar.

Now to identify the module we first decompose it into a direct sum of irreducible K -modules. We note that this is straightforward using weight vectors. Then we find an isomorphism to W by acting with elements of I , as in the discussion above. Again we have that such an isomorphism is unique up to scalar multiplication.

Checking the hypothesis of Lemma 3.2. The group $\text{Aut}_0(\mathbb{Y})$ is connected and has dimension 6. It suffices to prove that the automorphism group of $L_0(\mathbb{Y})$ is also connected and 6-dimensional.

Any automorphism of $L_0(\mathbb{Y})$ is also an automorphism of the 3-dimensional radical J and an automorphism of the two-dimensional nilradical I . The group of automorphisms of I is $\mathrm{GL}_2(F)$, which is connected of dimension 4. The subgroup of automorphisms of J fixing I pointwise is isomorphic to F^2 : an element in $x \in J - I$ can be mapped to any element in y iff their adjoint action in I is the same, and this is true iff $x - y \in I$. Hence the group of automorphisms of J is of dimension 6 and connected. Finally, we show that any automorphism ϕ of J can be extended in a unique way to an automorphism ψ of $L(\mathbb{Y})$. Let $x \in J - I$ arbitrary. There is a unique Levi subalgebra R that normalizes x . The automorphism ψ , if exists, has to send R to the unique subalgebra R' that normalizes $\phi(x)$. For any $y \in R$, there is a unique element $y' \in R'$ such that $[y, z] = [y', z]$ for all $z \in I$. We set $\psi(y) := y'$, and this determines the isomorphism ψ uniquely. - It follows that $\mathrm{Aut}(L_0(\mathbb{Y}))$ is isomorphic to $\mathrm{Aut}(J)$, hence it is also 6-dimensional and connected. \square

Remark 3.7 The algorithm for identifying \mathbb{Y} does not require factorization of polynomials or solving nonlinear equations; field arithmetic and solving linear systems are sufficient. Hence the result—in particular whether L is isomorphic to $L_0(\mathbb{Y})$ or not—does not change when we extend the field F . We rediscovered the well-known fact that there are no proper twists of \mathbb{Y} (see [7]).

Timings. We tried our algorithm on examples which we constructed from the canonical surface (given by the binomial ideal with 20 generators) by a linear transformation of the projective space. The randomly generated matrix of the transformation has integral entries with the given maximal absolute value (the first column in Table 2). We see that almost the whole time is spent for finding the Lie algebra of the surface.

4 Parametrizing Twists of $\mathbb{P}^1 \times \mathbb{P}^1$

The only Del Pezzo surfaces over algebraically closed fields are $\mathbb{P}^1 \times \mathbb{P}^1$ and \mathbb{Y} . Hence any Del Pezzo surface over F is a twist of one of these two. There are no proper twists of \mathbb{Y} by remark 3.7, but we still have to deal with proper twists of $\mathbb{P}^1 \times \mathbb{P}^1$. (We will see that some of them do have a parametrization.)

Here is a theorem that says that many twists do not have a parametrization.

perturb	eqns max	LA size	prm size	time	LA time
1	4	10	46	4.43	4.23
5	85	47	211	21.25	20.76
10	280	59	266	28.21	27.58
20	912	72	327	35.94	35.16
50	6372	93	424	51.66	50.43
100	26625	103	475	58.08	56.84
200	98407	127	584	69.29	67.69
500	599186	145	666	82.81	80.89
1000	1926906	159	724	91.26	89.11
2000	7973589	179	819	101.04	98.50
5000	60259495	207	957	118.99	115.94
10000	246171712	219	1008	129.49	126.24

Description of the columns: as in Table 1.

Table 2: Parametrizing \mathbb{Y} .

Theorem 4.1 *Assume that $X \cong C_1 \times C_2$, where C_1 and C_2 are twists of \mathbb{P}^1 . Then X has a parametrization only if $C_1 \cong \mathbb{P}^1$ and $C_2 \cong \mathbb{P}^1$.*

Proof. Assume that X has a parametrization. Then it has in particular an F -rational point $p \in X(F)$. The two projections give F -rational points $\pi_1(p) \in C_1(F)$ and $\pi_2(p) \in C_2(F)$. Because a twist of \mathbb{P}^1 with an F -rational point is already isomorphic to \mathbb{P}^1 , it follows that $C_1 \cong \mathbb{P}^1$ and $C_2 \cong \mathbb{P}^1$. \square

By Theorem 4.1, we can restrict our attention to varieties that are not products. But how is this reflected in the Lie algebra? Here is the answer to this question.

Theorem 4.2 *A twist of $\mathbb{P}^1 \times \mathbb{P}^1$ is a product of two twists of \mathbb{P}^1 iff its Lie algebra is a direct sum of two twists of \mathfrak{sl}_2 .*

Proof. “Only if”: if $X \cong C_1 \times C_2$, then $\text{Aut}_0(X)$ is the direct product of the two normal subgroups $\text{Aut}(C_1)$ and $\text{Aut}(C_2)$. It follows that $L_0(X) = L_0(C_1) \oplus L_0(C_2)$.

“If”: assume that X is not a product. Let E be a Galois extension of F with the property that $X_E \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then $\text{Pic}(X_E) \cong \mathbb{Z}^2$, and the divisor classes $(1, 0)$ and $(0, 1)$ define the two projections to \mathbb{P}^1 . We claim that the Galois group G interchanges these two classes. Indeed, the action

of G is \mathbb{Z} -linear, preserves the intersection product and the canonical class $(-2, -2)$, and this shows that $(1, 0)$ can only be mapped to itself or to $(0, 1)$. If $(1, 0)$ was fixed, then the G -orbit sum of some divisor $D \in \text{Div}(X_E)$ such that $[D] = (1, 0)$ would be in $(|G|, 0) = |G|(1, 0)$, and since it is in $\text{Div}(X)$, it would then define a projection to a twist of \mathbb{P}^1 , contradicting our assumption that X is not a product.

Since G interchanges the two classes defining the two projections, it also interchanges the two normal subgroups of $\text{Aut}_0(X_E)$ of dimension 3, and hence it also interchanges the two ideals of $L_0(X, E)$. It follows that these ideals are not defined over F , hence $L_0(X, F)$ is simple. \square

For any $a \in F^* - (F^*)^2$, we will now construct a twist S_a of $\mathbb{P}^1 \times \mathbb{P}^1$, called *sphere*, which is not a product, in the simplest possible way. More precisely, let E be the quadratic field extension $F[\alpha]/(\alpha^2 - a)$. Then $(S_a)_E \cong \mathbb{P}^1 \times \mathbb{P}^1$.

The construction works as follows. We start with the anticanonical embedding of $\mathbb{P}_E^1 \times \mathbb{P}_E^1 \subset \mathbb{P}_E^8$. We label coordinates and unit vectors in E^9 by ordered pairs of integers in $\{0, 1, 2\}$. The surface $\mathbb{P}^1 \times \mathbb{P}^1$ is embedded by mapping $((s : 1), (t : 1))$ to the point with coordinates $x_{ij} = s^i t^j$ with respect to the basis e_{ij} for $i, j = 0, 1, 2$.

Let σ be the generator of the Galois group $G := \mathcal{G}(E, F)$. Then σ induces an F -linear map $\Sigma : E^9 \rightarrow E^9$ defined by $ce_{ij} \mapsto \sigma(c)e_{ji}$. Obviously Σ preserves $\mathbb{P}_E^1 \times \mathbb{P}_E^1$. Similarly as in [5], the involution Σ defines an F -structure on $\mathbb{P}_E^1 \times \mathbb{P}_E^1$. We set S_a to be the F -variety defined by this structure. The set of F -rational points on $\mathbb{P}_E^1 \times \mathbb{P}_E^1$ is equal to the set of E -rational points fixed under Σ .

The variety S_a is not a product because two factors in $\mathbb{P}^1 \times \mathbb{P}^1$ are interchanged by Σ , hence none of the two projection morphisms is defined over F .

Let V be the F -linear subspace of E^9 of fixed vectors. By Galois descent, $\dim(V) = 9$; we give the explicit basis

$$B := \{e_{00}, e_{11}, e_{22}, e_{01} + e_{10}, e_{12} + e_{21}, e_{02} + e_{20}, \\ \alpha^{-1}(e_{10} - e_{01}), \alpha^{-1}(e_{21} - e_{12}), \alpha^{-1}(e_{20} - e_{02})\}.$$

We can give a parametrization of S_a in the coordinates with respect to the basis B in the parameters $u := \frac{1}{2}(s + t)$ and $v := \frac{\alpha}{2}(s - t)$, namely

$$(1 : P : P^2 : u : Pu : 2u^2 - P : v : vP : 2uv),$$

where $P = u^2 - a^{-1}v^2 = st$.

4.1 Yet Another Identification Problem

In this subsection, we give an algorithm for solving the embedded identification problem for S_a . We denote its Lie algebra $L_0(S_a, F)$ by \mathfrak{s}_a . We will show that it is the F -linear space of elements in $\mathfrak{sl}_2(E) \oplus \mathfrak{sl}_2(E)$ that are fixed under the semilinear automorphism that exchanges two fixed Chevalley bases of the two summands and takes the coefficients to their conjugates.

Of course, the algorithm first needs to find an $a \in F^*$ such that the given surface X is isomorphic to S_a , if exists.

The *centroid* $\Gamma(L)$ of a Lie algebra L is the centralizer of $\text{ad } L$ in $\mathfrak{gl}(L)$. It is easy to check that the centroid of \mathfrak{s}_a is isomorphic to $E := F[\alpha]/(\alpha^2 - a)$, the field extension defined by a .

Proposition 4.3 *Let X be a twist of $\mathbb{P}^1 \times \mathbb{P}^1$ which is not a product. Then the centroid E of $L_0(X, F)$ is a quadratic field extension of F , and X_E is a product.*

Proof. By Theorem 4.2, we can assume that $L := L_0(X, F)$ is simple. By [6], Theorem 10.1, the centroid of a simple Lie algebra is a field. Because $\Gamma(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2)$ has dimension 2, and the dimension of the centroid does not change when we extend the field, it follows that $E = \Gamma(L)$ is a quadratic field extension. Because $\Gamma(L \otimes_F E) = \Gamma(L) \otimes_F E = E \otimes_F E$ is not a field, it follows that $L \otimes_F E$ is not simple. \square

Of course, proposition 4.3 solves the subtask of finding a . We just have to compute the centroid. Once we have a , there is of course still no guarantee that X is isomorphic to S_a ; the following proposition decides this.

Proposition 4.4 *Let X be a twist of $\mathbb{P}^1 \times \mathbb{P}^1$ which is not a product. Let $E := F[\alpha]/(\alpha^2 - a)$ be the centroid of $L_0(X, F)$. Then the following are equivalent.*

- a) *The varieties X and S_a are isomorphic.*
- b) *The Lie algebras $L_0(X, F)$ and \mathfrak{s}_a are isomorphic.*
- c) *The varieties X_E and $\mathbb{P}^1 \times \mathbb{P}^1$ are isomorphic over E .*
- d) *The Lie algebras $L_0(X, E)$ and $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ are isomorphic over E .*

Proof. (a) \implies (c): by construction, $(S_a)_E = \mathbb{P}^1 \times \mathbb{P}^1$.

(c) \implies (d) (and also (a) \implies (b)) are obvious.

(d) \implies (b): in the following by σ -semilinear homomorphism f of Lie algebras L, L' we mean an F -linear Lie algebra homomorphism such that

$f(cv) = \sigma(c)f(v)$ for every $c \in E$ and $v \in L$. The Galois automorphism σ induces a σ -semilinear Lie algebra homomorphism σ_L on $L_0(X, E) = L_0(X, F) \otimes_F E$ which fixes $L_0(X, F)$. By assumption, $L_0(X, E)$ is isomorphic to $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, hence it is a sum of two ideals L_1 and L_2 , each isomorphic to $\mathfrak{sl}_2(E)$. The automorphism σ_L interchanges L_1 and L_2 , because otherwise both would be fixed under the Galois action and $L_0(X, F)$ would not be simple. Let us fix a Chevalley basis in $\mathfrak{sl}_2(E)$ and let $\sigma_{\mathfrak{sl}_2}$ be the σ -semilinear automorphism of \mathfrak{sl}_2 fixing this basis. Let $\psi : L_1 \rightarrow \mathfrak{sl}_2(E)$ be a Lie algebra isomorphism. We define the E -linear Lie algebra isomorphism $\varphi : L_1 \oplus L_2 \rightarrow \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ componentwise by sending $x_1 \oplus x_2$ to $\psi(x_1) \oplus (\sigma_{\mathfrak{sl}_2} \circ \psi \circ \sigma_L)(x_2)$. Let $\Sigma : \mathfrak{sl}_2 \oplus \mathfrak{sl}_2(E) \rightarrow \mathfrak{sl}_2 \oplus \mathfrak{sl}_2(E)$ be the semilinear automorphism that interchanges the Chevalley bases of the summands, i.e. $\Sigma(x_1 \oplus x_2) = \sigma_{\mathfrak{sl}_2}(x_2) \oplus \sigma_{\mathfrak{sl}_2}(x_1)$. Then the two semilinear Lie algebra homomorphisms $\Sigma \circ \varphi$ and $\varphi \circ \sigma_L$ from $L_0(X, E)$ to $\mathfrak{sl}_2(E) \oplus \mathfrak{sl}_2(E)$ coincide. It follows that the restriction of φ to $L_0(X, F)$ (as the subset of $L_0(X, E)$ which is fixed under σ) is a Lie algebra isomorphism to the subset of $\mathfrak{sl}_2(E) \oplus \mathfrak{sl}_2(E)$ the image of which is fixed under Σ , and this is \mathfrak{s}_a .

(b) \implies (a): the Lie algebra \mathfrak{s}_a acts on F^9 in two ways, namely as the Lie algebra of S_a , and via the Lie algebra isomorphism to $L_0(X, F)$ which we assume to exist. Over E , these two Lie modules are both isomorphic to the unique irreducible module with highest weight $(2, 2)$ (see subsection 3.2). In particular, they are isomorphic to each other. The matrix of a module isomorphism describes also a Lie algebra isomorphism by conjugation. Therefore it is a solution to a linear system and hence defined over F . Then by Section 3 the claim follows. \square

Here is the identification algorithm for S_a applied to a given twist X of $\mathbb{P}^1 \times \mathbb{P}^1$ such that the centroid of $L_0(X, F)$ is $E := F[\alpha]/(\alpha^2 - a)$.

1. Decompose $L_0(X, E)$ into $L_1 \oplus L_2$, using the algorithm described in [4].
2. Construct a Lie algebra isomorphism $\psi : L_1 \rightarrow \mathfrak{sl}_2(E)$, using the algorithm described in subsection 3.1. If the two Lie algebras are not isomorphic, then X is not isomorphic to S_a .
3. Construct a Lie algebra isomorphism $\varphi : L_0(X, F) \rightarrow \mathfrak{s}_a$ by restricting the E -isomorphism $\psi \oplus (\sigma\psi\sigma) : L_1 \oplus L_2 \rightarrow \mathfrak{sl}_2(E) \oplus \mathfrak{sl}_2(E)$ to $L_0(X, F)$.
4. Construct a Lie module isomorphism M between the two \mathfrak{s}_a -modules

given by the action on S_a and by the Lie algebra isomorphism φ . Return M .

The correctness of the algorithm follows from the following statements.

- Theorem 4.2 and Proposition 4.3 together imply that $L_0(X, E)$ decomposes into two ideals.
- Proposition 4.4 implies that $L_1 \cong \mathfrak{sl}_2(E)$ is necessary for X being isomorphic to S_a .
- The proof of Proposition 4.4, implication (d) \implies (b), shows that the construction in step 3 is indeed a Lie algebra isomorphism (hence $L_1 \cong \mathfrak{sl}_2(E)$ is also sufficient for X being isomorphic to S_a).
- The proof of Proposition 4.4, implication (b) \implies (a), shows that the module isomorphism exists, is unique up to scalar multiplication, and takes S_a into X .

Timings. For testing the algorithm we constructed examples as follows. We have chosen $d \in \mathbb{Z}$ such that $d \notin \mathbb{Q}^2$ (given in the first column of Table 3). Then the sphere in \mathbb{P}^3 given by $z_0^2 - z_1^2 = z_2^2 - dz_3^2$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{Q}(\sqrt{d})$ but not over \mathbb{Q} . We anticanonically embedded the sphere over \mathbb{Q} into \mathbb{P}^8 obtaining such a surface described by 14 binomials and 6 polynomials with 4 terms. Afterwards we made a linear transformation similar to the two previous cases, just here the generated matrix is sparser, to obtain examples solvable in practice. Since we have to identify two \mathfrak{sl}_2 's over $\mathbb{Q}(\sqrt{d})$, we have to solve two relative norm equations. This is very time consuming, therefore we were able to parametrize only “small” examples.

4.2 Completeness of the Method

Assume that X is a twist of $\mathbb{P}^1 \times \mathbb{P}^1$, which is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and not isomorphic to S_a for any $a \in F^*$. We distinguish two cases.

1. Assume that X is a product. Then X does not have a parametrization by Theorem 4.1.

discr	perturb (sparse)	eqns max	LA size	prm size	time	LA time	normeq time
-1	1	3	3	9	2.460	0.670	1.010
3	1	5	3	23	3.620	1.030	1.700
8	1	15	5	1135	211.340	1.270	123.230
-1	2	10	5	92	41.690	1.250	38.670

discr – square of the primitive element used for the construction,
normeq time – the time (in sec) needed for solving two relative norm equations (is
a part of “time”).

Description of the other columns: as in Table 1.

Table 3: Parametrizing the sphere.

2. Assume that X is not a product. Let E be the centroid of $L_0(X, F)$, which is a quadratic field extension by Proposition 4.3. By Proposition 4.4, X_E is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. On the other side, X is a product by Proposition 4.3. Then X does not have a parametrization over E by Theorem 4.1. Consequently X does not have a parametrization over F .

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